Near-Optimum Gram-Schmidt Conjugate Direction Signal Detection for Massive MIMO

Juan Minango[1] and Celso de Almeida[1]

State University of Campinas, Campinas SP, Brazil,
(jminango, celso)@decom.fee.unicamp.br

Abstract. In massive multiple-input multiple-output (M-MIMO) systems, when the number of base station (BS) antennas is much higher than the number of users, linear minimum mean-square error (MMSE) detector is able to achieve a near-optimum performance once the M-MIMO channel presents the property of asymptotic orthogonality. But, MMSE detector involves matrix inversion that presents high complexity especially when the number of users increases. In this paper, to avoid the matrix inversion, we propose a novel near-optimum signal detector with low-complexity, which is based on the Gram-Schmidt conjugate direction (GSCD) method, which reduces the complexity from $O(K^3)$ to $O(K^2)$, where K is the number of users. Numerical results reveal that the proposed detector has fast convergence and achieves the near-optimum performance of MMSE detector with a small number of iterations.

Keywords: Massive MIMO, Gram-Schmidt conjugate direction, matrix inversion, MMSE detector, BER.

1 Introduction

Massive multiple-input multiple-output (M-MIMO) systems with hundreds of base station (BS) antennas serving simultaneously dozen of users in the same bandwidth are considered as an emerging technology for the fifth generation (5G) cellular networks, due to its high spectral efficiency [8, 1]. Theoretical results have demonstrated that M-MIMO systems can provide high peak data rates and increase the spectral efficiency [6, 10]. However, M-MIMO systems faces several challenging problems in practice. One of which is the practical signal detector in the uplink.

The optimum signal detector is the maximum likelihood (ML), but it finds difficulties in practical M-MIMO implementations due to its high complexity [5], which is exponential with the number of users. On the other hand, linear minimum mean square error (MMSE) detector is able to achieve the near-optimum performance in M-MIMO systems due to the asymptotic orthogonality channel matrix property [11]. But, MMSE detector involves matrices inversion, whose complexity is cubic with respect to the number of users. To reduce the matrix inversion complexity, the authors of [13] have proposed to obtain an approximate matrix inversion through the Neumann Series (NS) algorithm. But, NS suffers from significant performance loss when M-MIMO scales up with a marginal complexity reduction [7].

In this paper, the Gram-Schmidt conjugate direction (GSCD) signal detection for M-MIMO systems is proposed, which avoids the computation of the matrix inversion and it is superior over NS approach. Besides, we analyze the complexity and demonstrate that the proposed detector has quadratic complexity with respect to the number of users. Finally, numerical results verify that the proposed detector is able to achieve the near-optimum performance of MMSE detector with a small number of iterations.

The remainder of this paper is organized as follows. Section 2 introduces the uplink of a M-MIMO system and the MMSE detector. The proposed detector is presented in Section 3. The numerical results of the bit error rate (BER) performance together with the complexity are shown in Section 4. Finally, the conclusions are drawn in Section 5.

Notation: Vectors and matrices are represented by lower and upper case boldface, respectively, $\mathbf{a}$ and $\mathbf{A}$. $(\cdot)^H$, $(\cdot)^{-1}$, $|\cdot|$ and $\|\cdot\|$ are conjugate transpose, matrix inversion, absolute value, and matrix norm, respectively. $\mathbf{I}_N$ is the $N \times N$ identity matrix.

2 System Model

In this paper, the uplink of a M-MIMO system is considered, which employs M antennas at the BS to simultaneously serve K single-antenna users, where $M \gg K$ [9]. The $M \times 1$ received vector $\mathbf{y}$ at the BS antennas is given by:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{x}$ denotes the $K \times 1$ transmitted vector coming from K different users, $\mathbf{H}$ is the $M \times K$ flat Rayleigh fading channel matrix, whose entries consists of independent and identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance, and $\mathbf{n}$ denotes the $M \times 1$ additive white Gaussian noise (AWGN) vector, whose elements consists of i.i.d complex Gaussian random variables with zero mean and variance σ_n^2 by each element. The detection vector of $\mathbf{x}$ can be obtained by MMSE detector as [11]:

$$\hat{\mathbf{x}} = \mathbf{W}^{-1}\tilde{\mathbf{y}}, \tag{2}$$

where $\tilde{\mathbf{y}} = \mathbf{H}^H\mathbf{y}$ is the matched-filter output of $\mathbf{y}$, and the MMSE filtering matrix $\mathbf{W}$ is given by:

$$\mathbf{W} = \mathbf{H}^H\mathbf{H} + \sigma_n^2\mathbf{I}_U. \tag{3}$$

From (2), the computation of $\mathbf{W}^{-1}$ given by (3) requires cubic complexity with respect to the number of users, that is $O(K^3)$. Thus, the increase of K in M-MIMO systems leads to larger matrices $\mathbf{W}$ that have expensive costs of inversion.

Recently, [13] has proposed the NS algorithm to replace the matrix inversion computation with NS expansion. Thus, by using the following decomposition

$\mathbf{W} = \mathbf{D} + \mathbf{F}$, where $\mathbf{D}$ and $\mathbf{F}$ are the diagonal and the off-diagonal matrices of $\mathbf{W}$, respectively, the NS expansion to compute $\mathbf{W}^{-1}$ can be written as [13]:

$$\mathbf{W}^{-1} \approx \sum_{k=1}^{\infty} \left(-\mathbf{D}^{-1}\mathbf{F}\right)^{k-1} \mathbf{D}^{-1}. \tag{4}$$

For $k > 3$, the NS algorithm complexity is $O(K^3)$, which shows that in this case none complexity reduction is achieved.

3 Gram-Schmidt Conjugate Direction (GSCD) Detector

From (2), the MMSE detector can be interpreted as solving the following linear equation [11]:

$$\mathbf{W}\hat{\mathbf{x}} = \tilde{\mathbf{y}}, \tag{5}$$

which can be solved in an iterative way [11, 2].

It has proved in [11] that $\mathbf{W}$ given by (3) is an Hermitian $\left(\mathbf{W}^H = \mathbf{W}\right)$ and positive-definite ($\mathbf{x}^H\mathbf{W}\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x}$) matrix. Thus, by using these two properties, a novel near-optimum signal detector which is based on the Gram-Schmidt conjugate direction (GSCD) is proposed. Basically, the GSCD detector is an efficient iterative algorithm with low-complexity that minimizes the following quadratic function [4]:

$$\phi\left(\hat{\mathbf{x}}\right) = \frac{1}{2}\hat{\mathbf{x}}^H\mathbf{W}\hat{\mathbf{x}} - \hat{\mathbf{x}}^H\tilde{\mathbf{y}}, \tag{6}$$

where it is easy to show that the minimum value of $\phi\left(\hat{\mathbf{x}}\right)$ is $\tilde{\mathbf{y}}^H\mathbf{W}^{-1}\tilde{\mathbf{y}}/2$, achieved by setting $\hat{\mathbf{x}} = \mathbf{W}^{-1}\tilde{\mathbf{y}}$. Thus, minimizing $\phi\left(\hat{\mathbf{x}}\right)$ given by (6) and solving (5) are equivalent problems.

The GSCD detector computes $\hat{\mathbf{x}}$ iteratively, where each iteration has low-complexity. The main advantages of GSCD detector is that it converges after K iterations. However, the GSCD detector can be stopped earlier while still obtaining a signal detection close to the exact one. This leads to a low-complexity, as an alternative to compute the matrix inversion of $\mathbf{W}$. Algorithm 1 summarizes the proposed detector.

From algorithm 1, on line 4, we first compute the matched-filter output $\tilde{\mathbf{y}}$ and the MMSE filtering matrix $\mathbf{W}$. Then, a rough initial solution $\hat{\mathbf{x}}_0$ is obtained by exploiting the diagonal dominant nature of $\mathbf{W}$ in M-MIMO systems [11], that is given by $\hat{\mathbf{x}}_0 = \mathbf{D}^{-1}\tilde{\mathbf{y}}$ (see line 5). Obviously, the complexity to invert $\mathbf{D}$ is very small. Later, on line 6, we initialize the residual error vector $\mathbf{r}_0$ and the direction vector $\mathbf{d}_0$ used in the GSCD iterative procedure.

On lines 8-14, the GSCD iterative procedure is employed, where a variant of the well-known Gram-Schmidt orthogonalization procedure [4] is used to find the $\mathbf{W}$-orthogonal search direction vector $\mathbf{d}_k$, where $\mathbf{W}$-orthogonality means that $\mathbf{d}_i^H\mathbf{W}\mathbf{d}_j = 0$, $\forall j < i$. Once a suitable $\mathbf{d}_k$ is found, a step in that direction is taken as $\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + \beta_k\mathbf{d}_k$ (see line 10), whose residual error vector

Algorithm 1 The GSCD signal detector

```
1: input:
2:     H and y
3: initialization:
4:     ỹ = H^H y, W = H^H H + σ_n^2 I_K and D = diag(W)
5:     ŝ_0 = D^{-1} ỹ,
6:     r_0 = ỹ − W ŝ_0, and d_0 = r_0
7:     k = 0
8: while k ≤ K do
9:     β_k = (d_k^H r_k) / (d_k^H W d_k)
10:    x̂_{k+1} = x̂_k + β_k d_k
11:    r_{k+1} = r_k − β_k W d_k
12:    d_{k+1} = r_{k+1} − Σ_{i=1}^{k} (r_{k+1}^H W d_i) / (d_i^H W d_i) d_i
13:    k = k + 1
14: end while
15: output:
16:    ŝ_K
```

$\mathbf{r}_k$ is minimal. Then, the next search direction vector $\mathbf{d}_{k+1}$ is computed by applying the Gram-Schmidt orthogonalization step to $\mathbf{r}_k$ (see line 12). Note that $\mathbf{d}_{k+1}$ is linearly independent from all the other previous search direction vectors. Thus, the sequence $\{\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_K\}$ is generated, where $\hat{\mathbf{x}}_K$ is the decision of the transmitted vector $\mathbf{x}$ after K iterations (see line 16).

Since the K search direction vectors $\{\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_K\}$ span the whole vector space, the GSCD iterative procedure always converges at most in K iterations [4]. However, usually the GSCD iterative procedure can be terminated before K iterations.

3.1 Convergence rate

It has been demonstrated in [12] that the convergence rate of GSCD iterative procedure mainly depends on the condition number of $\mathbf{W}$. Since $\mathbf{W}$ is an Hermitian positive-definite (HPD) matrix, which means that the condition number of $\mathbf{W}$ is given by:

$$\kappa = \frac{\lambda_{\max}(\mathbf{W})}{\lambda_{\min}(\mathbf{W})}, \tag{7}$$

where $\lambda_{\max}(\mathbf{W})$ and $\lambda_{\min}(\mathbf{W})$ represent the largest and the smallest eigenvalues of $\mathbf{W}$, respectively. Thus, if we assume that $\hat{\mathbf{x}}_K$ is the accurate final solution of $\hat{\mathbf{x}}$, then, the error at the k-th iteration can be denoted as:

$$\left\|\hat{\mathbf{x}}_K - \hat{\mathbf{x}}_k\right\|_{\mathbf{W}} \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \left\|\hat{\mathbf{x}}_K - \hat{\mathbf{x}}_0\right\|_{\mathbf{W}}, \tag{8}$$

where $\|\mathbf{q}\|_{\mathbf{W}} = \sqrt{\mathbf{W}\mathbf{q}\cdot\mathbf{q}}$.

From (8), we note that the convergence rate increases as the condition number of $\mathbf{W}$ approaches to one due to the error at the k-th iteration is principally influenced by the parameter $2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k$.

Besides, note that to compute (8), it is necessary to know a priori $\lambda_{\max}(\mathbf{W})$ and $\lambda_{\min}(\mathbf{W})$ which is difficult in practice. However, by taking into consideration the asymptotic orthogonality channel matrix property, a tight upper bound of (8) can be obtained as we show in Lemma 1.

Lemma 1. *In M-MIMO systems, the error produced by GSCD detector at the k-th iteration can be upper bounded by:*

$$\|\hat{\mathbf{x}}_K - \hat{\mathbf{x}}_k\|_{\mathbf{W}} \leq 2\left(\frac{1}{\sqrt{\alpha}}\right)^k \|\hat{\mathbf{x}}_K - \hat{\mathbf{x}}_0\|_{\mathbf{W}}, \tag{9}$$

where $\alpha = M/K$.

Proof. Since $\mathbf{W}$ follows a Wishart distribution. Let α being the loading factor. As M and K grow, the largest and the smallest eigenvalues of $\mathbf{W}$ converge, respectively, to [3]:

$$\lambda_{\max}(\mathbf{W}) \to M\left(1+\frac{1}{\sqrt{\alpha}}\right)^2 \quad \text{and} \quad \lambda_{\min}(\mathbf{W}) \to M\left(1-\frac{1}{\sqrt{\alpha}}\right)^2. \tag{10}$$

Substituting (10) into (7) and applying it into (8), we have that $\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right) \leq \left(\frac{1}{\sqrt{\alpha}}\right)$, consequently the error produced by GSCD detector at the k-th iteration is upper bounded by (9), which completes the proof.

From Lemma 1, we can note that when α increases, (9) decreases, thus, a faster convergence rate is achieved. Therefore, GSCD detector is ideal for M-MIMO signal detection. Fig. 2 compares the theoretical and upper bound of the error given by (8) and (9), respectively, against α. We note that the upper bound is very tight especially when α increases.

4 Simulation Results

4.1 BER performance

In order to evaluate the performance of the proposed detector, the simulation results of the BER against the average received signal-to-noise ratio (SNR) are present, which are compared with the recently NS iterative detector [13]. Besides, the BER of MMSE detector employing matrix inversion is also included as benchmark. Two typical M-MIMO systems with $M \times K = 128 \times 16$ and $M \times K = 256 \times 16$ are considered. The 64-QAM modulation scheme is employed on both M-MIMO systems. Besides, we assume that the channel has Rayleigh fading components which are perfectly known at the BS.

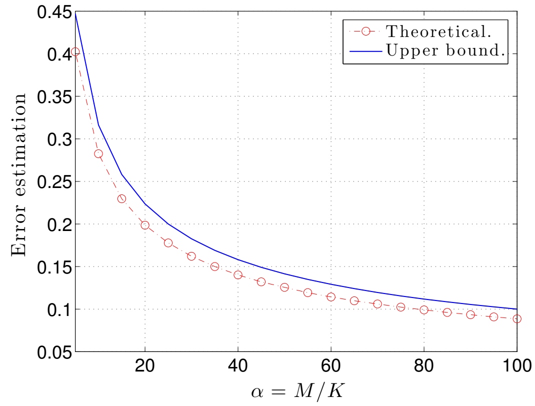


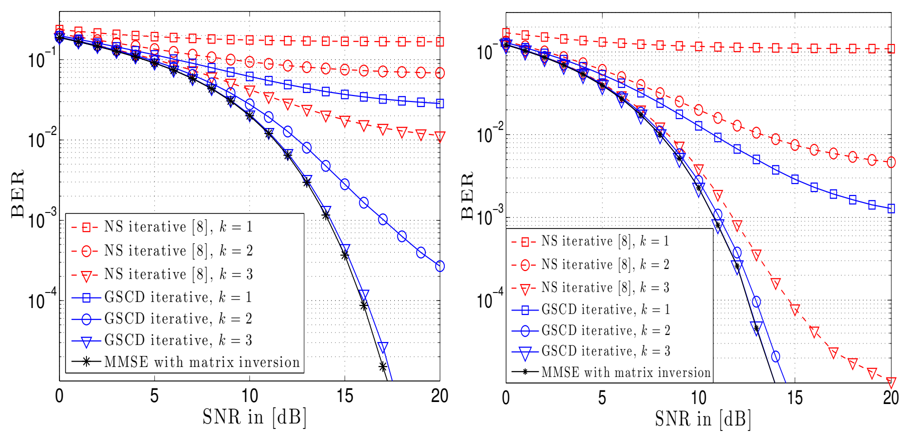
Fig. 1: Comparison between the theoretical and upper bound of the error against the loading factor $\alpha = M/K$.

Fig. 2a shows the BER comparison between NS and GSCD iterative detectors for the $M \times K = 128 \times 16$ M-MIMO system where k denotes the number of iterations. From this figure, it is evident that the BER of both detectors improves with the number of iterations. However, for same k, the GSCD detector outperforms NS detector, which reveals that a faster convergence rate is achieved by the proposed detector. Observe that with only 3 iterations, the performance loss of GSCD detector compared to MMSE detector is within 0.1 dB for a $BER = 10^{-4}$. Thus, GSCD detector is able to achieve the near-optimum performance of MMSE detector.

Fig. 2b compares the BER between NS and GSCD iterative detector when the M-MIMO system is $M \times K = 256 \times 16$. By comparing Fig. 2a and Fig. 2b, we note that when M increases, a better BER performance is obtained by both detectors. For example, when $k = 3$, for the $M \times K = 128 \times 16$ M-MIMO system, NS detector presents a BER floor at 10^{-2}, while for the $M \times K = 256 \times 16$ M-MIMO system, it achieves a BER floor at 10^{-5}. Note that NS detector does not converge in both M-MIMO systems. In contrast, when $k = 3$, the BER loss of GSCD detector compared to MMSE detector is within 0.1 and 0.01 dBs for a $BER = 10^{-4}$ for the $M \times K = 128 \times 16$ and $M \times K = 256 \times 16$ M-MIMO system, respectively. This shows that the convergence rate of GSCD detector is more robust with respect to the M-MIMO scales. Furthermore, given the same number of iterations, its superiority grows as the loading factor α increases.

4.2 Computational complexity

To analyze the complexity of GSCD detector, we use the term "flop" to mean a multiply-add operation. Since the matched-filter output $\tilde{\mathbf{y}}$ and the MMSE filtering matrix $\mathbf{W}$ need be computed both by MMSE detector given by (3) and by the proposed detector (see line 4), we only focus on the complexity of the



(a) (b)

Fig. 2: BER as a function of SNR for M-MIMO systems with (a) $M \times K = 128 \times 16$, (b) $M \times K = 256 \times 16$, where k denotes the number of iterations.

GSCD iterative procedure. Thus, from algorithm 1, we clearly notice that the proposed detector (lines 8-14) involves matrix-vector multiplications. Therefore, the required total number of flops is:

$$C = k\left[K^2 + 2K + k\left(2K + 1\right)\right] - K \cong O\left(kK^2\right), \tag{11}$$

where k is the number of iterations.

Table 1: Computational complexity comparison

Number of iterations	NS iterative algorithm [13]	Proposed detector based on GSCD method
$k = 1$	$2K$	$K^2 + 3K + 1$
$k = 2$	$12K^2 - 4K$	$2K^2 + 11K + 4$
$k = 3$	$8K^3 + 4K^2 - 2K$	$3K^2 + 23K + 9$
$k = 4$	$16K^3 - 4K^2$	$4K^2 + 39K + 16$

Table 1 presents the complexity in terms of flops for both NS [13] and GSCD detector. It is well known that the complexity of MMSE detector with matrix inversion is $O(K^3)$. On the other hand, Table 1 shows that NS detector can reduce the complexity from $O(K^3)$ to $O(K^2)$ when $k \leq 2$. However, when $k \geq 3$ the complexity is $O(K^3)$, which indicates that none complexity gain can be achieved in relation to MMSE detector. Furthermore, Table 1 shows that the complexity of GSCD detector is $O(kK^2)$ for any arbitrary number of iterations k.

5 Conclusion

In this paper, due to the MMSE filtering matrix in M-MIMO system is Hermitian and positive-definite, we have proposed a novel low-complexity signal detector for M-MIMO uplink, which approach is based on the Gram-Schmidt conjugate direction (GSCD). We have shown that GSCD detector can reduce the complexity from $O(K^3)$ to $O(kK^2)$. Numerical results reveal that GSCD detector achieves the near-optimum performance of MMSE detector with a small number of iteration. In addition, GSCD detector outperforms the Neumann series detector in terms of performance and complexity.

References

1. Araújo, D.C., Maksymyuk, T., de Almeida, A.L.F., Maciel, T., Mota, J.C.M., Jo, M.: Massive MIMO: Survey and Future Research Topics. IET Commun. **10**(15) (2016) 1938–1946
2. Bjorck, A.: Numerical Methods for Least Squares Problems. Society for Industrial and Applied Mathematics (1996)
3. Edelman, A.: Eigenvalues and Condition Numbers of Random Matrices. SIAM J. Matrix Anal. Appl. **9**(4) (1988) 543–560
4. Golub, G.H., Van Loan, C.F.: Matrix Computations (3rd Ed.). Johns Hopkins University Press, Baltimore, MD, USA (1996)
5. Higashinaka, M., Motoyoshi, K., Nagayasu, T., Kubo, H., Shibuya, A., Okazaki, A.: Likelihood Estimation for Reduced-Complexity ML Detectors in a MIMO System. In: IEEE 65th Veh. Techn. Conf. (Apr. 2007) 2104–2108
6. Hoydis, J., ten Brink, S., Debbah, M.: Massive MIMO in the UL/DL of Cellular Networks: How Many Antennas Do We Need? IEEE J. Sel. Areas Commun. **31**(2) (Feb. 2013) 160–171
7. Kang, B., Yoon, J.H., Park, J.: Low Complexity Massive MIMO Detection Architecture Based on Neumann Method. In: Int. SoC Design Conf. (ISOCC). (Nov. 2015) 293–294
8. Lu, L., Li, G.Y., Swindlehurst, A.L., Ashikhmin, A., Zhang, R.: An Overview of Massive MIMO: Benefits and Challenges. IEEE J. Sel. Topics Signal Process. **8**(5) (Oct. 2014) 742–758
9. Marzetta, T.L.: Noncooperative Cellular Wireless with Unlimited Numbers of Base Station Antennas. IEEE Trans. Wireless Commun. **9**(11) (Nov. 2010) 3590–3600
10. Prasad, K.N.R.S.V., Hossain, E., Bhargava, V.K.: Energy Efficiency in Massive MIMO-Based 5G Networks: Opportunities and Challenges. IEEE Wireless Commun. **24**(3) (June 2017) 86–94
11. Rusek, F., Persson, D., Lau, B.K., Larsson, E.G., Marzetta, T.L., Edfors, O., Tufvesson, F.: Scaling Up MIMO: Opportunities and Challenges with Very Large Arrays. IEEE Signal Process. Mag. **30**(1) (Jan. 2013) 40–60
12. Shewchuk, J.R.: An Introduction to the Conjugate Gradient Method Without the Agonizing Pain. Technical report, Pittsburgh, PA, USA (1994)
13. Zhu, D., Li, B., Liang, P.: On the Matrix Inversion Approximation Based on Neumann Series in Massive MIMO Systems. In: IEEE Int. Conf. Commun. (ICC). (Jun. 2015) 1763–1769