An Interval Insight to Adomian Decomposition Method for Ordinary Differential Systems by Considering Uncertain Coefficients with Chebyshev Polynomials

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Abstract- Generally, parameters in the mathematical models of engineering problems are considered deterministic. Although, in practice, there are always some uncertainties in the model parameters. Uncertainty can make an accurate or even wrong representation for the analyzed model. there is a wide reason which cause the uncertainties, like: measurement error, inhomogeneity of the process, etc. This problem leads researches to analyze the problem from a different point of view. When the uncertainty is present in the process, traditional methods of exact values can't solve the problem with no inaccuracies and mistakes. Interval analysis is a method which can be utilized to solve these kind of problems. In this paper, an interval Adomian decomposition method combined with Chebyshev polynomial is introduced. The proposed interval Adomian method is then validated through ODE systems. The simulation results are applied on 4 practical case studies and the results are compared with interval Euler and Taylor methods. Final results show that the proposed methodology has a good accuracy to find the proper interval and to effectively handle the wrapping effect to sharpen the range of non-monotonic interval.

Keywords-Interval Analysis; Adomian decomposition method; Chebyshev polynomials; Ordinary differential equations; Uncertainty.

1. Introduction

Generally, during the mathematical modelling of a practical phenomenon, the corresponding parameters have considered as exact values. However, the parameters of these phenomena have some uncertainties. These uncertainties can be generated from different reasons like neglecting some nonlinear terms on the model, simplifications and etc. These uncertainties lead the researcher to solve problems in a wrong way and consequently, the final result will be wrong. There are different ways to illustrate these uncertainties [1]. Uncertainties can be modeled by probabilistic variables, fuzzy variables, interval variables, etc. But the most proper method is to use the interval arithmetic. In the interval arithmetic, uncertainties stand throughout a definite lower and upper bounds. In other words, although the uncertainties quantity is unknown, but an interval can be defined to them. Ordinary differential equations (ODEs) include a wide range of applications like the systems modelling, optimal control, etc. There are different techniques which are introduced to solve these types of systems.

In the recent decades, decomposition methods have been shown as an effective, easy, and accurate methods to solve a great deal of linear and nonlinear, ordinary, partial, deterministic or stochastic differential equations by approximation. They have also rapid convergence to achieve accurate solutions [2-5].

Among these methods, Adomian decomposition method has been transformed to a popular technique for solving functional differential equations like ordinary differential equations [6], differential-algebraic equations [7], non-linear fractional differential equations [8], delay differential equations [9], etc.

From the above, it is clear that the Adomian decomposition method is a proper method for solving the ODE systems. Now what happened if these systems have some uncertainties. In this study, an improved adomian decomposition method is introduced to achieve a proper and robust solution. The main idea is to find a proper interval bound which keeps the system stable even if the parameters are changed in the considered interval uncertainty.
we also benefits from the Chebyshev orthogonal polynomial for simplify the complicated source terms to achieve more compressed solution rather than the Taylor series.

2. Interval arithmetic

When we a mathematical model of an engineering system is build, there are always some simplifications; although simplification reduces the system complication, but it makes some natural uncertainties on the model. In other words, some uncertain coefficients are appeared in the model [10]. Hence, utilizing normal methods for modelling or solving these types of systems cause some problems. However, an uncertainty coefficient has unknown quantity, but it is bounded and can be considered in an interval. Interval arithmetic provides a set of methods to keep track these uncertainties during the computations [11]. The interval set for an interval number can be described as:

$$X = [\underline{x}, \bar{x}] = \left\{x \mid x \in \mathbb{R} \cup \{-\infty, \infty\}, \underline{x} \le x \le \bar{x}\right\}, \qquad (1)$$

$$I(\mathbb{R}) = \left\{X \mid X = [\underline{x}, \bar{x}], \underline{x}, \bar{x} \in \mathbb{R}, \underline{x} \le \bar{x}\right\},$$

where, X defines an interval integer over $I(\mathbb{R})$ and $\underline{x}$ and $\bar{x}$ are the lower and upper bounds respectively. The midpoint value, the width of interval number and the radius of an interval can be defined as:

$$x_c = (\bar{x} + \underline{x})/2, \; x_r = x_w/2, \qquad (2)$$

$$x_w = \bar{x} - \underline{x} \qquad (3)$$

The basic interval arithmetic operations are described so that the interval guarantees the reliability of interval results. The main interval arithmetic operations between two interval numbers:

$$X + Y = [\underline{x} + \underline{y}, \bar{x} + \bar{y}] \qquad (4)$$

$$X - Y = [\underline{x} - \bar{y}, \bar{x} - \underline{y}] \qquad (5)$$

$$X \times Y = [\min\{\underline{x}\underline{y}, \bar{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\bar{y}\}, \; \max\{\underline{x}\underline{y}, \bar{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\bar{y}\}] \qquad (6)$$

$$X / Y = X \times 1/Y, \qquad (7)$$
$$1/Y = [1/\bar{y}, 1/\underline{y}], \; 0 \notin [\underline{y}, \bar{y}]$$

$$X^n = \begin{cases} \left[0, \max(\underline{x}^n, \bar{x}^n)\right] & n = 2k, \; 0 \in [x] \\ \left[\min(\underline{x}^n, \bar{x}^n), \max(\underline{x}^n, \bar{x}^n)\right] & n = 2k, \; 0 \notin [x] \\ [\underline{x}^n, \bar{x}^n], & n = 2k+1 \end{cases} \qquad (8)$$

The interval function F is an inclusion function of if $f \; \forall X \in I(\mathbb{R}), \; f(F) \subseteq F(X)$.

The main objective of this study is to find an interval function *F* from *f* to achieve an interval form of our method.

3. Chebyshev based Adomian Decomposition Method

In the Adomian decomposition method, the unknown function (i.e. $y(x)$) is decomposed into an infinite series $y(x) = \sum_{i=0}^{\infty} y_i(x)$ where $y_0, y_1, \ldots$ are evaluated recursively. It is important to know that if the function has nonlinearity ($N(y(\mathrm{x}))$), it should be obtained by the following equation:

$$N(y(x)) = \sum_{n=0}^{\infty} A_n \qquad (9)$$

where, $A_n = A_n(y_0(x), y_1(x), \ldots, y_n(x))$ are the Adomian polynomials:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty} \lambda^i y_i(x)\right)\Bigg|_{\lambda=0} \qquad n = 0, 1, 2, \ldots \qquad (10)$$

Consider an ordinary differential equation as follows:

$$Ly + Ry + Ny = g(x), \tag{11}$$

here, N describes the nonlinear operator, L defines the highest invertible derivative, R is the linear differential operator less order than L and g represents the source term. By applying the inverse term "L^{-1}" into the expression $Ly = g - Ry - Ny$, we have:

$$y = g + f - L^{-1}(Ry) - L^{-1}(Ny), \tag{12}$$

Where the function f describes the integration of the source term and g is the given conditions. By considering the last equation, the recurrence relation of y can be simplified as follows:

$$\begin{cases} y_0 = g + f, \\ y_1 = -L^{-1}(Ry_0) - L^{-1}(Ny_0) \\ \vdots \\ y_{k+1} = -L^{-1}(Ry_k) - L^{-1}(Ny_k), \quad k \geq 0 \end{cases} \tag{13}$$

Adomian decomposition theoretical convergence can be found in [12]. If the series converges to the considered solution, then

$$y = \lim_{M \to \infty} \tilde{y}_M(x), \tag{14}$$

where, $\tilde{y}_M(x) = \sum_{i=0}^{M} y_i(x)$ [2].

In [13], a new improved version of the decomposition method is introduced using Chebyshev approximation method. The illustrated method has overcome to the Taylor series in accuracy to expand the source term function. The advantage of the modified approach is verified through several illustrative examples. Since, in this paper, we expand the source term in Chebyshev series:

$$g(x) \approx \sum_{i=0}^{M} a_i T_i(x), \tag{15}$$

where $T_i(x)$ represents the first kind Chebyshev polynomial and can be evaluated as follows:

$$\begin{aligned} &T_0(x) = 1, \\ &T_1(x) = x, \\ &T_{k+1}(x) = 2x T_k(x) - T_{k-1}(x), \quad k \geq 1 \end{aligned} \tag{16}$$

Since, we have:

$$\begin{cases} y_0 = g + L^{-1}\left(\sum_{i=0}^{n} a_i T_i(x)\right), \\ y_{k+1} = -L^{-1}(Ry_k) - L^{-1}(Ny_k), \quad k \geq 0 \end{cases} \tag{17}$$

4. Interval Adomian Decomposition with Uncertainty

Consider the following ordinary differential equation (ODE) with considered interval initial conditions:

$$\begin{aligned} &F\left(X, Y', \ldots, Y^{(n)}, \Delta\right) = 0, \\ &Y\left(x_0\right) = Y_0, \ldots, Y^{(n)}\left(x_0\right) = Y_n, \end{aligned} \tag{18}$$

Where $\Delta = \left(d_1, d_1, d_2, d_3, \ldots, d_m\right)^T$ is an uncertain parameters and $d_k \in [\underline{d}_k, \overline{d}_k], k = 0, 1, \ldots, (n+1)$.

The main purpose of this paper is to introduce an interval version of Adomian decomposition method for solving the ODE problems with uncertainties. To do this, let consider a standard form as below:

$$\begin{aligned} &\underbrace{dY^{(n)} + dY'' + \ldots + dY' + dY}_{R\{y(x)\}} + aN\{Y(X)\} = bG(X), \\ &a \in [\underline{a}, \overline{a}], \quad b \in [\underline{b}, \overline{b}], \end{aligned} \tag{19}$$

The first step is to convert the system into the uniform mode,

$$\underbrace{dY^{(n)} + dY'' + \ldots + dY' + dY}_{R\{y(x)\}} + \tilde{a}N\{Y(X)\} = \tilde{b}G(X), \tag{20}$$

Where the term $(\sim)$ shows the division of the coefficients with d_n. For instance, $\tilde{b}$ is $\left[\underline{b}, \overline{b}\right] / \left[d_n\right] = \left[\underline{b}, \overline{b}\right] \times \left[\frac{1}{\overline{d}_n}, \frac{1}{\underline{d}_n}\right]$.

d_n is assumed non-singular, i.e. $0 \notin d_n$. From the previous section, by using the given conditions we obtain:

$$Y = P_n + F - L^{-1}(RY) - L^{-1}(Ny), \tag{21}$$

Where, $P_n = [\underline{p}_n, \overline{p}_n]$ is an interval polynomial which has been achieved from the initial conditions as $P_n = Y_0 + Y_1 x + \ldots + Y_n x^n$. If the given conditions have the exact value, then $P_n = [p_n, p_n]$; these kinds of exact value in the interval analysis are so called degenerative intervals. The function F describes the interval integration of the source term. For computing the inverse operator of the source term we have:

$$F = [L^{-1}(bg(x))] = [\underline{b}, \overline{b}] \times [L^{-1}(\underline{g}(x)), L^{-1}(\overline{g}(x))], \tag{22}$$

By assuming $g = g_c + d_g$ where g_c and d_g comprise the certain and uncertain terms of g. Since,

$$\left[L^{-1}\left(\underline{g}(x)\right), L^{-1}\left(\overline{g}(x)\right)\right] = \left[L^{-1}\left(g_c(x) + \left[\underline{d}_g, \overline{d}_g\right]\right)\right] \tag{23}$$

Similar operation can be also utilized for achieving the linear and nonlinear differential operators. Since the total proposed Interval Adomian method can be formulated as follows:

$$\begin{cases} Y_0 = [g] + [\underline{b}, \overline{b}] \times [L^{-1}(\underline{g}(x)), L^{-1}(\overline{g}(x))], \\ Y_{k+1} = [-L^{-1}(R\underline{y}_k) - L^{-1}(N\underline{y}_k), -L^{-1}(R\overline{y}_k) - L^{-1}(N\overline{y}_k)], \quad k \geq 0 \end{cases} \tag{24}$$

And the source term can be achieved as:

$$\begin{aligned} G(x) &= \sum_{i=0}^{n} [a_i][T_i(x)], \\ &= \left[\underline{b}, \overline{b}\right] \times L^{-1}\left(\sum_{i=0}^{n} [\underline{a}_i][\underline{T}_i(x)], \sum_{i=0}^{n} [\overline{a}_i][\overline{T}_i(x)]\right), \end{aligned} \tag{25}$$

5. Illustrative examples

To demonstrate the effectiveness of the proposed method, we give four different examples of linear and nonlinear ordinary differential equations. The algorithms are performed by Matlab 2013a.

5.1. Case study 1

Consider for $-1 \leq x \leq 1$

$$Y''(x) - [d]Y(x) = 0, \quad [d] \in [1,2]$$

$$Y([0]) \in [1], \; Y'([0]) \in [b] = [-0.1, 0.1],$$

The problem above shows a differential equation where uncertainty in the coefficient term($[d]$), the initial condition $[b]$ is also uncertain and appeared the only thing we know is that it stands in an interval.

The purpose of the solution is to find a region which includes all different values within the represented interval. According to the formula,

$$LY + [d]RY = 0$$

We can defined that $L = \frac{d^2}{dx^2}$, $\breve{R} = [d]Y(x)$, $NY = G(x) = 0$. By applying the inverse operator $L^{-1} = \int_0^x \int_0^x \{.\}dxdx$ into the main equation,

$$Y(x) = [1,1] + [-0.1, 0.1]x + [d]\int_0^x \int_0^x \sum_{n=0}^{\infty} Y_n(x)\,dx\,dx,$$

and finally the recurrence relation in below can be utilized to achieve the $Y(x)$:

$$Y_0(x) = [1] + [b]x,$$

$$Y_1(x) = [d]\int_0^x\int_0^x Y_0(x)\,dx\,dx = [1,2]\frac{x^2}{2!} + [-0.2, 0.2]\frac{x^3}{3!},$$

$$Y_2(x) = [d]^2\int_0^x\int_0^x Y_1(x)\,dx\,dx = [1,4]\frac{x^4}{4!} + [-0.4, 0.4]\frac{x^5}{5!},$$

$$\vdots$$

$$Y_n(x) = [d]^n\int_0^x\int_0^x Y_{n-1}(x)\,dx\,dx = [1,2^n]\frac{x^{2n}}{(2n)!} + 2^n[-0.1, 0.1]\frac{x^{(2n+1)}}{(2n+1)!}$$

Since, the final solution can be achieved by:

$$Y(x) = [\cosh(x)] + 0.1\left(x + 2\frac{x^3}{3!} + \ldots + 2^n\frac{x^{(2n+1)}}{(2n+1)!}\right)[-1,1].$$

By calculating the problem in the time interval between 0 and 2 and same step size h = 0.4, the minimum and maximum value of y at each step is obtained and given in the table 1. In this case, we also applied a random value in the considered interval and the results showed that the random solution is placed in the interval solution. We also compared the proposed method by the interval Euler [14] and Taylor methods [15]. Table 1 shows more details of this comparison. As it can be seen, the interval space in the proposed interval Adomian method achieves generally narrower interval than the others.

<Table 1 here>

5.2. Case study 2

In this example we consider a problem with more uncertainty both in the linear differential operator and the source term. Consider for $-1 \le x \le 1$

$$Y'(x) - [d_1]xY(x) = [d_2](6 - 3x^2),$$
$$Y(0) = 0,\quad 0.5 \le d_1 \le 1,\quad 0.5 \le d_2 \le 1.$$

Where $L = \frac{d}{dx}$, $\tilde{R} = [d_1]xY(x)$, $NY = 0$ and $\tilde{g}(x) = [d_2](6 - 3x^2)$. In this problem, we have:

$$Y(x) = 0 + [d_1]\int_0^x \sum_{n=0}^{\infty} xY_n(x)\,dx + [d_2]\int_0^x (6 - 3x^2)\,dx.$$

Now using the Adomian decomposition method we get,

$$Y_0(x) = [d_2](-x^3 + 6x) = \left[x^3/2 + 3x, -x^3 + 6x\right]$$

$$Y_1(x) = [d_1]\int_0^x xY_0(x)\,dx = ([d_1]\times[d_2])\int_0^x x(-x^3 + 6x)\,dx$$
$$= [0.25, 1]\left(-x^5/5 + 2x^3\right) = \left[0.05x^5 + 0.5x^3, -0.2x^5 + 2x^3\right]$$

$$Y_2(x) = [d_1]\int_0^x xY_1(x)\,dx = ([d_1]^2\times[d_2])\int_0^x x(-x^5/5 + 2x^3)\,dx$$
$$= [0.125, 1]\left(-0.03x^7 + 0.4x^5\right) = \left[0.004x^7 + 0.05x^5, -0.03x^7 + 0.4x^5\right]$$

$$\vdots$$

$$Y_n(x) = [d_1]^n\int_0^x xY_{n-1}(x)\,dx$$

So, we have:

$$Y(x) = [-0.03x^7 + 0.2x^5 + x^3 + 6x - + \ldots, -0.004x^7 + 3x - + \ldots].$$

<Table 2 here>

From Table 2, we can say that Interval Euler method fails the interval in the ”time=0.4". it is also obvious that the proposed interval Adomian method has narrower interval rather than the interval Taylor method.

5.3. Case study 3

Now, we consider a problem with complicated source term. Consider for $-1 \le x \le 1$,

$$Y''(x) - [d]Y(x) = e^x,$$
$$Y(0) = 1, Y'(0) = 1, \quad 0.5 \le d \le 1,$$

Where $L = \frac{d^2}{dx^2}$, $\tilde{R} = [d]Y(x)$, $NY = 0$, $g(x) = e^x$. By letting M=4 and using the Chebyshev approximation we have:

$$\mathrm{g}(x) = 0.0432x^4 + 0.1772x^3 + 0.4998x^2 + 0.9974x + 0.9892$$

Error ($\left|g(x) - e^x\right|$)for this approximation is 0.0006397 while using Taylor series give us an error about 0.0099. From the main equation, we have:

$$Y(x) = 1 + x - [d]\int_0^x\int_0^x \sum_{n=0}^{\infty} Y_n(x)dx\,dx + \int_0^x\int_0^x (e^x)dx\,dx$$

Now using the Adomian decomposition method we get,

$$Y_0(x) = 1 + x + \int_0^x\int_0^x g(x)dx\,dx = 1 + x + 0.4946x^2 + 0.1662x^3 + 0.0417x^4 + 0.0089x^5 + 0.0014x^6$$

$$Y_1(x) = -[d]\int_0^x\int_0^x Y_0(x)dx\,dx = [-2.5e^{-5}x^8 - 2.119e^{-4}x^7 - 0.0014x^6 - 0.0083x^5 - 0.0412x^4 - 0.1667x^3 - 0.5x^2,$$
$$-1.25e^{-5}x^8 - 1.059e^{-4}x^7 - 0.0007x^6 - 0.00415x^5 - 0.0206x^4 - 0.0833x^3 - 0.25x^2],$$

$$\vdots$$

$$Y_n(x) = [d_1]' \int_0^x xY_{n-1}(x)dx,$$

And finally with using $\tilde{y}_M(x) = \sum_{i=0}^{M} y_i(x)$ the solution has been achieved.

<Table 3 here>
<Fig.1 here>

from Table 3, it is obvious that the interval Euler and Taylor Methods fail their interval and don't include the random value, but the interval Adomian method includes the value.
we also apply Chebyshev approximation to the source term of the interval Adomian method. As it can be seen from the fig.1 (B), Chebyshev polynomial gives us tighter interval and from the interval arithmetic, it has better performance rather than the Taylor series. Furthermore, it is important to know that the sometimes lower and upper bound have crossover with each other in some ODEs. In this situation, we should consider the general bound in between them as the reliability region.

5.4. Case study 4

Consider the nonlinear ordinary differential equation for $0 \le x \le 1$

$$Y''(x) - [d_1]xY^2(x)Y'(x) - [d_2]Y^3(x) = e^x - 2e^{3x} - 3xe^{3x},$$
$$Y(0) = 1, \; Y'(0) = 1, \; 1 \le d_1, d_2 \le 2,$$

Where $L = \frac{d^2}{dx^2}$, $\tilde{R} = 0$, $N_1Y = Y^2(x)Y'(x)$, $N_2Y = Y^3(x)$, $g(x) = e^x - 2e^{3x} - 3xe^{3x}$.

By considering 4th order Chebyshev approximation to the source term,

$$g(x) \approx -32.612x^4 - 45.0884x^3 - 14.2512x^2 - 1.7918x + 7.1530$$

Since,

$$Y(x) = 1 + x + [d_1]\int_0^x \int_0^x x N_1(x)\,dx\,dx + [d_2]\int_0^x \int_0^x N_2(x)\,dx\,dx + \int_0^x \int_0^x g(x)\,dx\,dx$$

The Adomian polynomials for $N_1(x)$ and $N_2(x)$ are:

<Table 4>

So, Following the same illustrated approach,

$$Y_0(x) = 1 + x + 3.5765x^2 - 0.2986x^3 - 1.1876x^4 - 2.2844x^5 - 1.0871x^6$$

$$Y_1(x) = [d_1]\int_0^x \int_0^x Y_0^2(x) Y_0'(x)\,dx\,dx + [d_2]\int_0^x \int_0^x Y_0^3(x)\,dx\,dx = [x^2 + 2x^3 + 2.8x^4 + \ldots$$
$$, 2x^2 + 4x^3 + \ldots]$$

$$\vdots$$

$$y_n(x) = [d_1]\int_0^x \int_0^x Y_{n-1}^2(x) Y_{n-1}'(x)\,dx\,dx + [d_2]\int_0^x \int_0^x Y_{n-1}^3(x)\,dx\,dx$$

<Table 5 here>

Table 5 shows that interval Euler and taylor methods fail the interval from "Time=0.8”, but the proposed method is totally including the random value.

6. Conclusions

The interval Adomian decomposition is introduced for solving differential equations with uncertainties. This approach provides a robust approximation of the solution. The main advantage of this approach over traditional numerical methods is that the proposed method is the first time which is used the interval arithmetic to provide a robust result for ODEs with uncertain coefficients. In addition, for increasing the system accuracy, Chebyshev polynomial is utilized for expansion the source term.

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Tables:

Table 1. The region bound achieved by the interval Adomian method and a limited random input for case study 1 which is stand in the considered region.

Time	Interval Adomian Method	Interval Euler's Method [13]	Interval Taylor Method [14]	Random value
0	[1, 1]	[1, 1]	[1, 1]	1.00
0.4	[1.081, 1.164]	[1.02, 1.16]	[1.04, 1.21]	1.081
0.8	[1.337, 1.711]	[1.20, 1.68]	[1.25, 1.81]	1.337
1.2	[1.810, 2.819]	[1.57, 2.69]	[1.66, 3.01]	1.810
1.6	[2.576, 4.838]	[2.17, 4.48]	[2.34, 5.19]	2.577
2	[3.755, 8.378]	[3.09, 7.56]	[3.4, 9.08]	3.762

Table 2. The region bound achieved by the interval Adomian method and a limited random input for case study 2 standing in the considered region.

Time	Interval Adomian Method	Interval Euler's Method [13]	Interval Taylor Method [14]	Random value
0	[0, 0]	[0,1]	[0, 0]	0
0.4	[1.826, 2.466]	[2.23, 2.44]	[1.20, 2.46]	1.826
0.8	[3.850, 5.371]	[3.55, 5.26]	[2.40, 5.38]	3.850
1.2	[6.457, 9.323]	[4.98, 9.11]	[3.60, 9.54]	6.457
1.6	[10.271, 15.026]	[6.59, 15.42]	[4.80, 16.84]	10.271
2	[15.943, 22.743]	[8.51, 27.81]	[6.00, 32.52]	15.943

Table 3. The region bound achieved by the interval Adomian method and a limited random input for case study 3 standing in the considered region.

Time	Interval Adomian Method	Interval Euler's Method [13]	Interval Taylor Method [14]	Random value
0	[1, 1]	[1, 1]	[1, 1]	1
0.4	[1.112, 1.126]	[1.16, 1.50]	[1.21, 1.54]	1.112
0.8	[1.408, 1.439]	[1.73, 2.32]	[1.87, 2.44]	1.409
1.2	[1.934, 1.986]	[2.87, 3.64]	[3.20, 3.91]	1.934
1.6	[2.773, 2.858]	[4.91, 5.71]	[5.59, 6.23]	2.774
2	[4.070, 4.202]	[8.38, 8.92]	[9.70, 9.86]	4.071

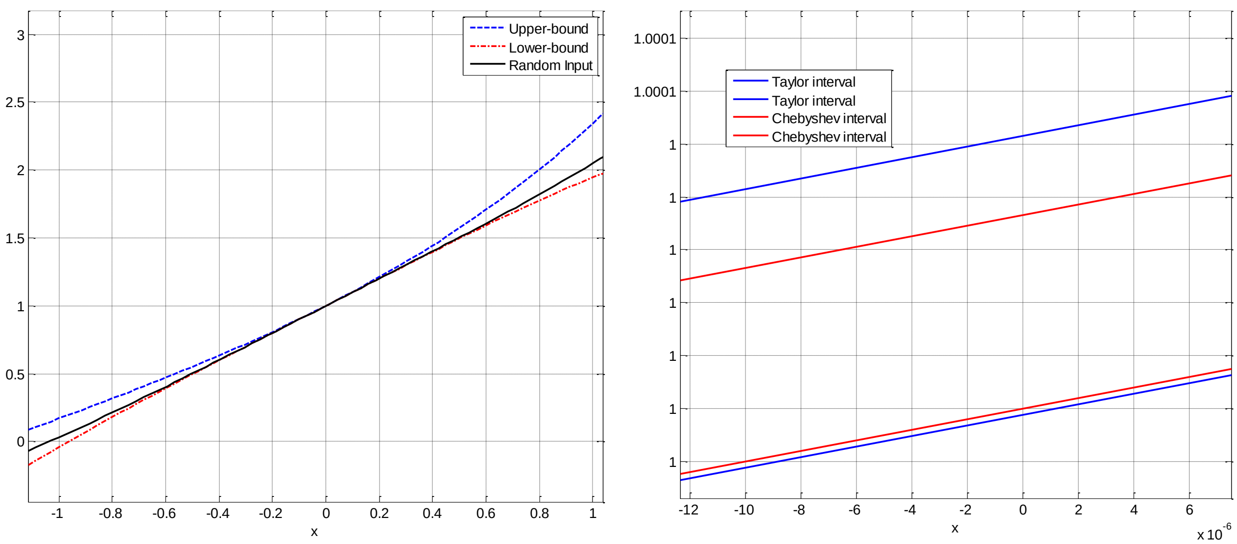
Table 4. Adomian polynomials for Nonlinear terms

$N_1(x)$	$N_2(x)$
$A_0(x) = Y_0^2 Y_0'$	$A_0(x) = Y_0^3$
$A_1(x) = Y_0^2 Y_1' + 2Y_0 Y_1 Y_0'$	$A_1(x) = 3Y_0^2 Y_1$
$A_2(x) = Y_0^2 Y_2' + 2Y_0 Y_1 Y_1' + 2Y_0 Y_2 Y_0'$	$A_2(x) = 3Y_0^2 Y_2 + 3Y_0 Y_1^2$
$\vdots$	$\vdots$

Table 5. The region bound achieved by the interval Adomian method and a limited random input for case study 4 standing in the considered region.

Time	Interval Adomian Method	Interval Euler's Method [13]	Interval Taylor Method [14]	Random value
0	[1, 1]	[1, 1]	[1, 1]	1
0.4	[1.398, 1.401]	[1.06, 1.37]	[0.99, 1.33]	1.399
0.8	[1.780, 1.819]	[0.27, 0.95]	[-0.67, 0.04]	1.781
1.2	[2.092, 2.307]	Divergent	Divergent	2.092
1.6	[2.234, 2.961]	Divergent	Divergent	2.235
2	[2.042, 3.941]	Divergent	Divergent	2.043

Figures:



(A) (B)

Fig.1. (A) The region bound achieved by the interval Adomian method and a limited random input which is stand in the considered region and (B) Comparison of the solution using Taylor and Chebyshev approximation for the case study 3.